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Schwarz–Christoffel Mappings: A General Approach

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A general method is developed for constructing equations of the Schwarz–Christoffel type. These equations define conformal transformations between regions of the complex plane, each of whose boundaries may consist of straight or continuously curved line segments and corners. The more familiar types of Schwarz–Christoffel equations are rederived, and some new types are obtained. © 1987 Academic Press, Inc

I. INTRODUCTION

Let $z = z(w)$ denote a one-to-one conformal map from region R_w of the "computational" plane, $w = u + iv$, to region R_z of the "physical" plane, $z = x + iy$. This is relevant to mesh-generation for study of flows in regions R_z of irregular boundary and to potential flow problems which may be simplified by reference to some standard region R_w with a more regular boundary.

The Schwarz–Christoffel equation (SCE), as customarily developed in texts[1,2] expresses dz/dw as a function of w then R_z is a polygon and R_w is a half-plane or disc. Davis [3] used a continuous limit of the polygon case to compute mappings when R_z is bounded by a more general curve. The SCE has also been adapted by Sridhar and Davis [4] to certain classes of channels. Floryan [5, 6] has applied SCEs to mesh calculations for regions bounded by walls, or within channels, including periodic and nonperiodic geometries.

The purposes of this paper are (a) to show how these SCE examples are special cases of a single formula of considerable generality and (b) to obtain additional examples and some improvement of existing ones for later applications to mesh generation and potential flow problems.

In terms of a suitably defined two-source Green's function on R_w, one can

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express an analytic function $S(w)$ in terms of the tangential derivative of its imaginary part along the boundary of R_w. If one sets $S(w) = z(w)$ or $z(w) - w$, the result is an integral equation with logarithmic kernel determining $z(w)$ when the boundary of R_z is known. Menikoff and Zemach [7] applied this formulation in a study of two-dimensional Rayleigh–Taylor instability.

Here, we set $S(w) = \log dz(w)/dw$ to obtain the generalized Schwarz–Christoffel equation. In this general equation, R_w may also have corners and curved boundary sections; in fact, the regions R_z, R_w, and their respective boundaries enter symmetrically. The class of geometries for which an SCE can be written explicitly corresponds to the class of R_w for which the requisite Green's function is obtainable in closed analytic form. The polygon formulas emerge as special cases of the formulas for general regions, rather than the other way around as in the Davis approach.

The derivation is given in the next section. Section III discusses some questions of implementation and particular cases. Section IV is a listing of SCEs for various R_w's for reference purposes. Alternative approaches to numerical conformal mapping are enumerated in Henrici's book [8].

II. Formulation

A. Boundaries

The boundary B_w of R_w may be composed of one or more rectifiable Jordan curves $B_w^{(i)}$, either closed and of finite length, or open and of infinite length. Let s_w label arc-length on each curve, measured from a suitable chosen reference point, with the direction of increasing s_w such that the neighboring part of R_w lies to the left (the "left-hand rule"). We suppose the boundary curves have a continuously turning tangent, except at a finite number of corners. Let $\theta_w(s_w)$ be the tangent's direction angle, oriented in direction of increasing s_w. We also suppose $d\theta_w/ds_w$ depends continuously on s_w, except at corners. A corner at s'_w will have a turning angle $\Delta\theta_w(s'_w)$:

$$\Delta\theta_w(s'_w) = \theta_w(s'_w + \varepsilon) - \theta_w(s'_w - \varepsilon).$$

Here and after, ε represents a positive infinitesimal going to zero. For each such corner, we may regard $d\theta_w/ds_w$ as having a Dirac δ-function contribution of $\Delta\theta_w(s'_w)\,\delta(s_w - s'_w)$.

For the R_z boundary, we adopt the same assumptions and similar notations: B_z, s_z, θ_z, $\Delta\theta_z$. We use $\theta_z[s_z]$ and $\theta_z(s_w)$ to indicate, respectively, dependence of θ_z on a point of B_z, and, through the mapping, on a point of B_w. If $z(s'_w)$ is a corner of B_z, then $d\theta_z/ds_w$ will include the term $\Delta\theta_z(s'_w)\,\delta(s_w - s'_w)$.

B. *The Log-Derivative Function* $S(w)$

Set $\log(dz(w)/dw) = S(w) = S_R(w) + iS_I(w)$. Then by our assumptions, $S(w)$ is analytic in R_w and has both tangential and normal derivatives on the boundary, except at corners. The left-hand rule and the Cauchy–Riemann conditions applied at a smooth boundary point imply

$$\frac{\partial S_R}{\partial n} = \frac{\partial S_I}{\partial s_w}, \qquad \frac{\partial S_I}{\partial n} = -\frac{\partial S_R}{\partial s_w}, \tag{2.1}$$

with $\partial/\partial n$ along the outward normal.

For dz and dw along their respective boundaries,

$$\log(dz) = \log|ds_z| + i\theta_z, \qquad \log(dw) = \log|ds_w| + i\theta_w,$$

so that, with w_B denoting a point of B_w,

$$S_R(w_B) = \mathrm{Re}(S(w_B)) = \log\left|\frac{dz}{dw}\right| = \log(ds_z/ds_w), \tag{2.2a}$$

$$S_I(w_B) = \mathrm{Im}(S(w_B)) = \theta_z - \theta_w, \tag{2.2b}$$

and ds_z/ds_w is always positive.

C. *Green's Functions*

For each component $B_w^{(i)}$ of B_w, let l_i be its length (which may or may not be infinite) and let β_i be an associated real constant. For the purposes of this analysis, we define the Green's function for R_w as a real function $G(w, w')$ satisfying

$$\nabla_w^2 G(w, w') = 2\pi\, \delta(u-u')\, \delta(v-v') \tag{2.3}$$

in $R(w)$, and satisfying

$$\frac{\partial}{\partial n} G(w_B, w') = \beta_i \tag{2.4}$$

for boundary points w_B on $B_w^{(i)}$. Integrating (2.3) over R_w and applying Green's theorem, we get

$$\sum_i \int_{B_w^{(i)}} \frac{\partial}{\partial n} G(w_B, w')\, ds_w(w_B) = 2\pi.$$

Therefore, by (2.4),

$$\sum_i \beta_i l_i = 2\pi, \tag{2.5}$$

showing that the β_i cannot all be chosen independently. Given a set of β_i consistent with (2.5), one may show that a Green's function $G(w, w')$ obeying the symmetry

$$G(w, w') = G(w', w) \tag{2.6}$$

exists, and is uniquely determined up to an additive constant. Hereafter, we assume $G(w, w')$ is symmetric.

Now set

$$g(w, w', w_0) = G(w, w') - G(w, w_0). \tag{2.7}$$

This is a "two-source" Green's function, obeying

$$\nabla_w^2 g(w, w', w_0) = 2\pi\, \delta(u-u')\, \delta(v-v') - 2\pi\, \delta(u-u_0)\, \delta(v-v_0) \tag{2.8}$$

and, for $w = w_B$ on any $B_w^{(i)}$,

$$\frac{\partial}{\partial n} g(w, w', w_0) = 0. \tag{2.9}$$

D. *The Generalized Schwarz–Christoffel Equation*

Multiply $\nabla^2 S_R(w) = 0$ by $g(w, w', w_0)$; multiply (2.8) by $S_R(w)$; subtract and integrate over R_w. Applying Green's theorem, (2.6), and (2.9), we can put the result in the form

$$\log\left|\frac{dz}{dw}\right| \equiv S_R(w) = S_R(w_0) - \frac{1}{2\pi}\int_{B_w} [G(w, w'_B) - G(w_0, w'_B)] \times \frac{\partial S_R}{\partial n}(w'_B)\, ds_w(w'_B). \tag{2.10}$$

If R_w has boundaries extending to ∞, interpretation of (2.10) requires consideration of the asymptotic behavior of G, S_R, $\partial G/\partial n$, and $\partial S_R/\partial n$; see the discussion in Section III.

In view of (2.1) and (2.2b), we put

$$\frac{\partial S_R(w_B)}{\partial n}\, ds_w(w_B) = \frac{\partial S_I(w_B)}{\partial s_w}\, ds_w(w_B) = dS_I(w_B)$$
$$= d\theta_z(s_w) - d\theta_w(s_w).$$

We have already recognized that θ_z and θ_w are continuously differentiable functions on B_w except that $\theta_w(s_w)$ has jumps at the corners of R_w, and $\theta_z(s_w)$ has jumps where the s_w are images, via the conformal map, of the corners of R_z.

Let $\Gamma(w,w'_B)$ be a complex extension of $G(w, w'_B)$. That is, Γ is an analytic function of w in R_w whose real part is $G(w, w'_B)$. In Section IV, Green's functions are given explicit representations in terms of logarithms of absolute values of analytic functions of w. Then the complex extensions may be obtained simply by dropping the absolute value signs. If the G's are replaced by Γ's in (2.10), the left side of (2.10) is an analytic function whose real part is S_R. Its imaginary part can only differ from S_I by a constant. Hence,

$$\log \frac{dz}{dw} = C_0 - \frac{1}{2\pi} \int_{B_w} [\Gamma(w, w'_B) - \Gamma(w_0, w'_B)] \times [d\theta_z(s'_w) - d\theta_w(s'_w)]. \tag{2.11}$$

Here, w_0 is an arbitrary point in R_w and C_0 is a complex constant, independent of w. Setting $w = w_0$ in (2.11), we see that $C_0 = \log dz/dw$, evaluated at w_0.

We call Eq. (2.11) the *generalized Schwarz–Christoffel equation with subtraction.* Subtraction refers to the differencing of $\Gamma(w, w'_B)$ and $\Gamma(w_0, w'_B)$. This goes back to the use of $g(w, w', w_0)$ rather than $G(w, w')$ in Green's theorem and serves two related purposes. First, the use of (2.9) rather than (2.4) simplifies the analysis. Second, when R_w extends to ∞, $g(w, w', w_0)$ decreases faster as $|w| \to \infty$ than $G(w, w')$, providing the extra margin of convergence of the integrals at $|w| \to \infty$ which is sometimes needed.

By specializing to w on B_w and taking the real part of (2.11), we have

$$\log\left(\frac{ds_z}{ds_w}\right) = \mathrm{Re}(C_0) - \frac{1}{2\pi} \int_{B_w} [G(w_B, w'_B) - G(w_0, w'_B)] \times [d\theta_z[s_z(s'_w)] - d\theta_w(s'_w)] \tag{2.12}$$

which provides the determining relation for $s_z(s_w)$ and $\theta_z(s_w)$ when R_z and B_z are given.

Simpler forms of (2.11) and (2.12) may be valid. If the integrals

$$\int \Gamma(w, w'_B)[d\theta_z - d\theta_w], \qquad \int \Gamma(w_0, w'_B)[d\theta_z - d\theta_w] \tag{2.13}$$

are separately convergent, then the second one may be lumped with C_0, giving, in terms of a new constant C, the *unsubtracted* form of the generalized Schwarz–Christoffel equation:

$$\log \frac{dz}{dw} = C - \frac{1}{2\pi} \int_{B_w} \Gamma(w, w'_B)[d\theta_z(s'_w) - d\theta_w(s'_w)]. \tag{2.14}$$

Note also that the portion

$$\int_{B_w} [\Gamma(w, w'_B) - \Gamma(w_0, w'_B)]\, d\theta_w(s'_w)$$

depends on R_w, but not on R_z; it can be evaluated prior to computation of the conformal map. For some R_w, including all those listed in Secton IV, its value is zero. But it may not vanish if, e.g., R_w has corners.

As a first illustration, let R_w be the upper half plane. Then $s'_w = u'$, $d\theta_w = 0$, $\Gamma(w, u') = 2\log(w - u')$, and (2.14) reduces to

$$\log \frac{dz}{dw} = C - \int_{-\infty}^{\infty} \log(w - u') \frac{d\theta_z(u')}{du'} \frac{du'}{\pi}. \tag{2.15}$$

In addition, suppose that R_z is a polygon with corners at $\{z_i\}$, which are images by the map of points $\{u_i\}$ on the real axis of the w-plane. Then,

$$\frac{d\theta_z(u')}{du'} = \sum_i \Delta\theta_z(u_i)\, \delta(u - u_i).$$

Also, the integrals (2.13) are indeed separately convergent because they are restricted to a finite interval, justifying the use of the unsubtracted SCE. Corresponding to the turning angles, $\Delta\theta_z(u_i)$ are the parameters $\alpha_i = -\Delta\theta_z(u_i)/\pi$. Equation (2.15) reduces to

$$\frac{dz}{dw} = e^C \prod_i (w - u_i)^{\alpha_i} \tag{2.16}$$

and so, the original equation of Schwarz and Christoffel is recovered in its primitive form.

III. Comment and Examples

A. Infinite Regions

The handling of regions R_z and R_w which have boundaries extending to infinity may be indicated by several examples.

First, suppose R_z is a semi-infinite region whose boundary extends to infinity at both ends and R_w is the upper half plane. Consider a restricted region R'_w which is a half-disc of radius ρ in the upper half plane with diameter on the u axis from $-\rho$ to ρ. We think of ρ as large and tending to ∞. Let R_z be similarly restricted to R'_z by some such "arc at ∞." Then the mapping of R'_w to R'_z has no difficulties at ∞. In a fully rigorous treatment, we might inquire how Green's function for the half-disc behaves as $\rho \to \infty$, but for present purposes, we shall suppose it is adequately represented by $G(w, w')$ for R_w, given in Section IV.B.1. Then G behaves like $\log|w|$ for large $|w|$, and (2.10) suggests, but does not assure, that $S_R = \log|dz/dw|$ also behaves like $\log|w|$. Assume, to be slightly more conservative, that S_R behaves no worse than $|w|^{1-\varepsilon}$ and $\partial S_R/\partial n$ behaves no worse than $|w|^{-\varepsilon}$. (See Section III.D for an example where S_R behaves like $(\log|w|)^2$). Also, $g(w, w', w_0)$ behaves like $|w|^{-1}$ and $\partial g/\partial n$ like $|w|^{-2}$. Then the "terms at ∞" in Green's theorem, involving

$$\int S_R\, \partial g/\partial n\, ds_w, \qquad \int g\, \partial S_R/\partial n\, ds_w$$

on the circumference of the half-disc decrease like $\rho^{-\varepsilon}$ or faster as $\rho \to \infty$ and disappear in this limit. Next, assume the "arc at ∞" of R'_z is to be mapped into the circumference of the half disc; the contribution of

$$\int g(w, w', w_0)[d\theta_z(s'_w) - d\theta_w(s'_w)]$$

to (2.10) vanishes as ρ^{-1}. This establishes the SCE for R_z and R_w subject to the assumed limit on the behavior of S_R at ∞.

Second, suppose R_z is the upper half plane and R_w is the interior of the unit disc about the origin, with $\Gamma(w, e^{i\phi'})$ the extended Green's function for R_w and ϕ' denoting the polar angle. Let R'_z be the rectangle with corners at $(L, 0)$, (L, iL), $(-L, iL)$, and $(-L, 0)$. Let these points be mapped into points on the unit disc with polar angles $\phi_1, \phi_2, \phi_3, \phi_4$. As $L \to \infty$, the interval $(\phi_1 \leqslant \phi \leqslant \phi_4)$ compresses to a single point ϕ_∞ such that $w \to e^{i\phi_\infty}$ as $|z| \to \infty$, and

$$\int_{\phi_1-\varepsilon}^{\phi_4+\varepsilon} \Gamma(w, e^{i\phi'})\, d\theta_z(\phi') \to \Gamma(w, e^{i\phi_\infty}) \int_{\phi_1-\varepsilon}^{\phi_4+\varepsilon} d\theta_z(\phi')$$
$$= \Gamma(w, e^{i\phi_\infty})\, \Delta(\infty),$$

where $\Delta(\infty)$ is the complete turning angle of B_z (in this case, $\Delta(\infty) = 2\pi$) as the tangent vector on B_z at $x \to +\infty$ turns around to align with the tangent vector at $x \to -\infty$. Thus in treating the complete integral

$$\int_{\phi'=-\pi}^{\pi} \Gamma(w, e^{i\phi'})\, d\theta(\phi'),$$

θ_z is counted as having a jump of $\Delta(\infty)$ associated with the point at ∞ of R_z, in addition to any other variation due to deviation of B_z from a straight line.

Third, suppose R_z has channels to ∞ bounded by straight lines with aperture angles of $\Delta(\infty)$ and $\Delta(-\infty)$ and this is to be mapped into a channel to ∞ in R_w bounded by parallel straight lines. See Fig. 1. For the Green's function integral, the regions can be restricted by arcs closing the channel openings, and then the arcs are allowed to recede to ∞. The total turning angles around the channel on the right are $\Delta(\infty) + \pi$ in R_z and π in R_w. Then, for the relevant part of the integral, we have

$$\int [\Gamma(w, w'_B) - \Gamma(w_0, w'_B)][d\theta_z(s'_B) - d\theta_w(s'_B)]$$

$$\to \lim_{w'_B \to \infty} [\Gamma(w, w'_B) - \Gamma(w_0, w'_B)]\, \Delta(\infty).$$

The bracket has a finite limit, namely $(\pi/2h)(w_0 - w)$. This is taken into account in Section IV.B.3, where the Green's function and the SCE for the horizontal channel in the w-plane, as illustrated in Fig. 1, are set forth.

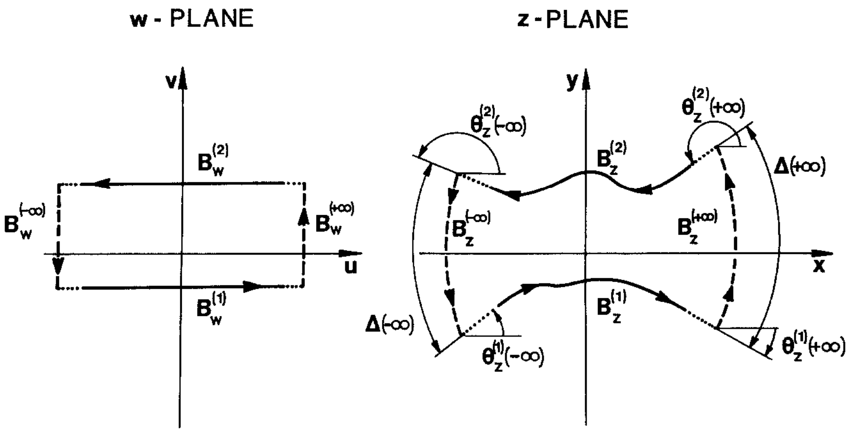

FIG. 1. Boundaries and turning angles for the conformal map of an irregular channel to a horizontal straight channel. The Green's function integrals over $B_w^{(\pm\infty)}$ give a finite contribution as these boundary segments recede to $\pm\infty$. The turning angles at infinity in the w-plane are $\theta_w^{(1)}(\pm\infty)=0$, $\theta_w^{(2)}(\pm\infty)=\pi$. See Section IV, Case 3.

B. *Flexibility in the Choice of Green's Functions*

An appropriate choice of Green's function can improve convergence of the integrals given in (2.13) at ∞. The $G(w, w')$ of the last section are not unique because the constants β_i of (2.4) need not be specified. To any specification of $G(w, w')$ can be added a harmonic, symmetric function of w, w' which obeys constant normal derivative boundary conditions, without changing the SCE formulas. For example, for a periodic map from the upper half w-plane (case 6 in the next section), both

$$G=\log|\sin\tfrac{1}{2}(w-w')|+\log|\sin\tfrac{1}{2}(w-w'^*)|$$

and

$$G=\log|e^{iw}-e^{iw'}|+\log|1-e^{i(w-w'^*)}|$$

are acceptable alternatives, but the second gives a more useful and more easily perceived behavior at ∞.

Also, the complex extension from $G(w, w_B)$ to $\Gamma(w, w_B)$ introduces choices. Taking the second G above in the limit $w'\to u'=\text{real}$, we can write, for example,

$$G(w,u')=2\log|1-e^{i(w-u')}| \qquad \text{or} \qquad G(w,u')=2\log|e^{iu'}-e^{iw}|.$$

Then "dropping the absolute value sign" leads to the alternatives

$$\Gamma(w,u')=2\log(1-e^{i(w-u')}) \qquad \text{or} \qquad \Gamma(w,u')=2\log(e^{iu'}-e^{iw}).$$

The first is analytically preferable because it vanishes like $O(e^{-v})$ as $v\to\infty$, and the log is made single-valued by the simple rule,

$$-\tfrac{1}{2}\pi\leqslant\arg(1-e^{i(w-u')})\leqslant\tfrac{1}{2}\pi, \qquad \text{for} \quad \operatorname{Im}(w)\geqslant 0. \tag{3.1}$$

For the next section, in cases 4 and 6, we also need the limit of the first $\Gamma(w, u')$ as $w \to u + i\varepsilon$,

$$\lim_{\varepsilon \to 0} 2 \log(1 - e^{i(u + i\varepsilon - u')})$$

$$= 2 \log \sin \tfrac{1}{2}(u - u') + i(u - u' - \pi) + 2 \log 2, \tag{3.2}$$

where

$$\sin \tfrac{1}{2}(u - u') \equiv \lim_{\varepsilon \to 0} \sin \tfrac{1}{2}(u - u' + i\varepsilon)$$

$$= \begin{cases} |\sin \tfrac{1}{2}(u - u')| & \text{for} \quad -\pi \leqslant u' < u \leqslant \pi, \\ |\sin \tfrac{1}{2}(u - u')| \, e^{i\pi} & \text{for} \quad -\pi \leqslant u < u' \leqslant \pi. \end{cases} \tag{3.3}$$

These relations are also relevant to IV.4.

C. *Symmetry of SCE between R_z and R_w*

Under a conformal map, the differential equations (2.3), (2.8), and the boundary condition (2.9) are invariant (though (2.4) is not). Thus a two-source Green's function $\bar{g}(z, z', z_0)$ for R_z is obtained by substituting the inverse of $z = z(w)$ in $g(w, w', w_0)$. If we put

$$\log(dz/dw) = -\log(dw/dz),$$

$$d\theta_z - d\theta_w = -(d\theta_w - d\theta_z)$$

into the SCE for the map from R_w to R_z, and take account of the invariance of the two-source Green's function, we get the SCE for the map of R_z to R_w. In other words, the SCE is symmetric in form under interchange of the z and w planes.

D. *On the Subtracted and Unsubtracted Alternatives for the SCE*

The subtraced SCE, Eq. (2.11), is more general than the simpler, unsubtracted from Eq. (2.14) and is appropriate point of departure whenever there is some tedious analysis to be done on behaviors at infinity. But in many instances Eq. (2.11) reduces to Eq. (2.14).

We do not have rigorous necessary-and-sufficient conditions to characterize maps which require the subtracted SCE, but will offer some suggestive criteria and examples for mappings of half-infinite regions. In particular, the subtracted SCE is applicable in Cases 1 and 2 cited below in this subsection but fails in Case 3.

Let R_z be a half-infinite region bounded by a single curve and let R_w be the upper half plane. Write s for s_z and u for s_w with $s \to \pm\infty$ as $u \to \pm\infty$; $\theta_z[s]$ has limiting values θ_∞ and $\theta_{-\infty}$ for $s \to \pm\infty$. B_z will have a total turning angle $\Delta = (\theta_\infty - \theta_{-\infty})$ and a total turn parameter $\alpha = -\Delta/\pi$ subject to

$$-\pi \leqslant \Delta \leqslant \pi, \qquad -1 \leqslant \alpha \leqslant 1.$$

The Green's function with $w' = u'$ on the boundary is $2 \log |w - u'|$, and $d\theta_w = 0$, as noted in Section IV.1.

There seem to be three general cases for such mappings, if the boundary in the z-plane is reasonably smooth at large distances from the origin, as characterized below. In our rather general examples, the unsubtracted SCE holds if (1) the total turning angle of B_z is less than π or (2) the total turning angle equals π and the boundary curve approaches well-defined asymptotes at $\pm\infty$. But the unsubstracted SCE fails if (3) the turning angle is π and at least one arm of B_z fails to have an asymptote.

We consider only boundaries B_z such that the direction $\theta_z[s]$ approaches its asymptotic values smoothly according to some power law, specifically,

$$\theta_z[s] - \theta_\infty \sim s^{-a} \qquad \text{as} \quad s \to \infty, \tag{3.4a}$$

$$\theta_z[s] - \theta_{-\infty} \sim |s|^{-a'} \qquad \text{as} \quad s \to -\infty, \tag{3.4b}$$

where the powers a, a' are positive.

The condition that B_z has an asymptote at $s \to \infty$ is that

$$\lim_{s \to \infty} \int^s (\theta_z[s] - \theta_\infty)\, ds < \infty. \tag{3.5}$$

Thus, a boundary following the law (3.4a) approaches an asymptote if $a > 1$ and fails to have an asymptote if $0 < a \leqslant 1$. As examples, we note that for $s \to \infty$, $(\theta_z[s] - \theta_\infty)$ behaves like s^{-2} for a hyperbola, and like $s^{-1/2}$ for a parabola.

We now ask whether the integrals of (2.13) do converge separately under present circumstances. It is sufficient to consider the real parts of these integrals and to take $w = u =$ real. We shall analyze a more restrictive question, namely, whether, as $u \to \pm\infty$,

$$\int_{u'=-\infty}^{\infty} \log|u - u'|\, d\theta_z(u') \sim \log|u| \int d\theta_z(u')$$
$$\sim \log|u|\,(\theta_\infty - \theta_{-\infty}). \tag{3.6}$$

If so, the unsubtracted SCE holds, and for $|u| \to \infty$,

$$\log \frac{ds}{du} \sim C - (\theta_\infty - \theta_{-\infty})(\log|u|)/\pi \equiv C + \alpha \log|u|, \tag{3.7a}$$

so that

$$\frac{ds}{du} \sim e^C |u|^\alpha. \tag{3.7b}$$

We distinguish three cases:

Case 1. Suppose $\Delta < \pi$ so that $\alpha > -1$. Suppose $a > 0$, $a' > 0$. For large $|s|$, Eq. (3.7b) implies $|s| \sim |u|^{1+\alpha}$. Then, by (3.4)

$$d\theta_z(u) \sim u^{-1-a(1+\alpha)}\, du, \qquad u \to +\infty, \tag{3.8a}$$

$$\sim u^{-1-a'(1+\alpha)}\, du, \qquad u \to -\infty. \tag{3.8b}$$

The exponents of u in (3.8) are less than -1. Then one may verify that the large $|u|$ behavior of (3.6) is valid. Appendix A indicates how the algebra of this verification might go. Under the conditions of this case, Eqs. (3.6) and (3.7) are mutually consistent and the unsubtracted SCE holds.

As an example, let R_z be the exterior of the parabola

$$y^2 = 4p^2(x + p^2). \tag{3.9}$$

The conformal map is $z = (w + ip)^2$. One finds, following the left-hand rule, $\Delta = -\pi$, $\alpha = +1$, and for large $|s|$, $|s| \sim u^2$, $(\theta_z[s] - \theta_{\pm\infty}) \sim p\,|s|^{-1/2}$, $d\theta_z/du \sim pu^{-2}$. The unsubtracted SCE may be explicitly verified by elementary integration techniques.

Case 2. Suppose $\alpha = -1$ and Eqs. (3.4) hold, but with $a > 1$, $a' > 1$. For large $|s|$, we get from (3.7), $|s| \sim \log |u|$ so that

$$d\theta_z(u) \sim (\log |u|)^{-1-a}\, u^{-1}\, du, \qquad u \to +\infty, \tag{3.10a}$$

$$\sim \log |u|)^{-1-a'}\, |u|^{-1}\, du, \qquad u \to -\infty. \tag{3.10b}$$

Again Eqs. (3.6) and (3.7) are mutually consistent, and the unsubtracted SCE holds. The verification of (3.6) is not trivial; details are given in Appendix A.

As an example, let R_z be the interior of the right-hand loop of $y^2 = 1 - x^{-2}$. Then $\alpha = -1$, $s \sim \log |u|$, $|\theta_z(s) - \theta_{\pm\infty}| \sim |s|^{-3}$, $d\theta_z(u)/du \sim (\log |u|)^{-4}\, |u|^{-1}$, and the unsubtracted SCE holds for this case.

Case 3. Suppose $\alpha = -1$, but unlike Case 2, either $a \leqslant 1$, or $a' \leqslant 1$, or both. The rate at which $\theta_z(s) \to \theta_\infty$ as $s \to \infty$, or at which $\theta_z(s) \to \theta_{-\infty}$ as $s \to -\infty$, is no better than $O(|s|^{-1})$. Then $d\theta_z(u)/du$ vanishes no faster than $O((\log |u|)^{-2}\, |u|^{-1})$ and the integral of (3.6) does not exist. The unsubtracted SCE fails.

For example, let R_z be the interior of the parabola (3.9). The conformal map is

$$w = i \cosh(\pi \sqrt{z}/2p).$$

Then for large $|s|$, $|s| \sim \log^2 |u|$ and $|d\theta_z/du| \sim (\log |u|)^{-2}\, |u|^{-1}$. The integral of the subtracted SCE converges, but the integral of the unsubtracted SCE does not.

IV. A Short Catalog of Green's Functions and Schwarz–Christoffel Equations

A. *Discussion*

A Schwarz–Christoffel equation for a conformal mapping $z = z(w)$ between R_w and R_z has the general form

$$\log(dz/dw) = C + \sum_i Q_i, \tag{4.1}$$

where C is a complex constant and the sum is over Green's-function integrals on one or more segments $B_w^{(i)}$ of the boundary B_w of R_w.

When z and w are on their respective boundaries, the left side of (4.1) has real and imaginary parts as given by (2.2). At the start of a mapping calculation, the bounaries B_z and B_w are specified and the directions $\theta_z[s_z]$ and $\theta_w(s_w)$ of the tangents to these boundaries, respectively, are presumed known in terms of their respective arc-length variables. The real part of (4.1), with z and w restricted to their boundaries is then an integro-differential equation for the boundary map function $s_z = s_z(s_w)$ and the related angle function $\theta_z(s_w)$, $\theta_z(s_w) = \theta_z[s_z(s_w)]$. Depending on the geometry, the real or the imaginary parts of C, or both, may be fixed by initial data or by initial assumptions; otherwise, they must be solved for concurrently with $s_z(s_w)$ and $\theta_z(s_w)$. Once $\theta_z(s_w)$ is determined, Eq. (4.1) in its general form defines dz/dw throughout R_w and a subsequent integration yields $z = z(w)$. References [3–6] illustrate how this can be managed in particular cases.

Again depending on the geometry, Eq. (4.1) may not determine $z(w)$ uniquely, but may allow some map parameters to be preassigned, as is well known. The Riemann mapping theorem shows that in a mapping from one simply connected region to another, three real parameters may be preassigned. For example, these parameters could be the images of three boundary points. The map of a doubly connected region to an annular region allows one parameter, but the ratio of the annular radii must be determined as part of the map calculation. In Case 6 below, where B_z is periodic, it is natural to match the endpoints of a fundamental period of B_z with the points $u = -\pi$, $u = +\pi$ of B_w, and to let $z = \infty$ correspond to $w = \infty$; then no free parameters remain, and C is known in advance of the calculation.

The "unsubtracted" SCE, Eq. (2.14), suffices in Cases 4–10 (integration region bounded) and will suffice in Cases 1–3 if the boundaries to infinity are well behaved, e.g., if they meet the criteria of the first two cases of Section III.D.

B. *Mappings for Various Geometries*

Notation and conventions for the orientation of boundaries are given in Section II.A. Stars denote complex conjugation. The title of each listing below refers to the geometry of R_w.

1. *Upper half plane.*

R_w: $-\infty < u < \infty$, $0 < v < \infty$
B_w: $w_B = u$, $s_w = u$, $d\theta_w(s_w) = 0$

$$G(w, w') = \log|w - w'| + \log|w - w'^*|$$

$$\log(dz/dw) = C - \int_{u'=-\infty}^{+\infty} \log(w - u')\, d\theta_z(u')/\pi,$$

$$\operatorname{Im}(C) = \theta_z(\infty), \qquad 0 \leqslant \arg(w - u') \leqslant \pi.$$

2. *Lower half plane.*

R_w: $-\infty < u < \infty$, $-\infty < v < 0$

B_w: $w_B = u$, $s_w = -u$, $\theta_w(s_w) = 0$.

Write $\theta_z(u)$ for the B_z direction (rather than $\theta_z(-w)$).

$$G(w, w') = \log|w - w'| + \log|w - w'^*|$$

$$\log(dz/dw) = C - \int_{u'=+\infty}^{-\infty} \log(w - u')\, d\theta_z(u')/\pi,$$

$$\mathrm{Im}(C) = \theta_z(-\infty), \quad -\pi \leqslant \arg(w - u') \leqslant 0.$$

3. *Horizontal straight channel* (*see Fig.* 1).

R_w: $-\infty < u < \infty, 0 < v < h$

B_w: $w_B = u$, $s_w = u$, $\theta_w(s_w) = 0$ on $B_w^{(1)}$

$w_B = u + ih$, $s_w = -u$, $\theta_w(s_w) = \pi$ on $B_w^{(2)}$

$w_B = u + iv$, $s_w = v$ with $u \to \infty$, $0 \leqslant v \leqslant ih$ on $B_w^{(\infty)}$

$w_B = u + iv$, $s_w = -v$ with $u \to -\infty$, $0 \leqslant v \leqslant ih$ on $B_w^{(-\infty)}$.

The direction of $B_z^{(1)}$, $B_z^{(2)}$ are taken as $\theta_z^{(1)}(u)$, $\theta_z^{(2)}(u)$, respectively.

$$G(w, w') = \log|\sinh(\pi/2h)(w - w')| + \log|\sinh(\pi/2h)(w - w'^*)|$$

$$\log(dz/dw) = C + Q_1 + Q_2 + Q_\infty + Q_{-\infty},$$

$$Q_\infty + Q_{-\infty} = w[\theta_z^{(2)}(\infty) - \theta_z^{(1)}(\infty) + \theta_z^{(2)}(-\infty) - \theta_z^{(1)}(-\infty) - 2\pi]/(2h)$$

$$= w[\Delta(\infty) - \Delta(-\infty(]/(2h),$$

where $\Delta(\infty)$ and $\Delta(-\infty)$ are the aperture angles at the ends of the R_z channel, as shown in Fig. 1:

$$Q_1 = \int_{u'=-\infty}^{u'=+\infty} \log\left[\sinh\frac{\pi}{2h}(w - u')\right] d\theta_z^{(1)}(u')/\pi,$$

$$0 \leqslant \arg\left[\sinh\frac{\pi}{2h}(w - u')\right] \leqslant \pi,$$

$$Q_2 = \int_{u'=+\infty}^{u'=-\infty} \log\left[\sinh\frac{\pi}{2h}(w - u' - ih)\right] d\theta_z^{(2)}(u')/\pi,$$

$$-\pi \leqslant \arg\left[\sinh\frac{\pi}{2h}(w - u' - ih)\right] \leqslant \pi.$$

4. *Interior of unit disc.*

R_w: $w = \rho e^{i\phi}$, $0 \leqslant \rho < 1$, $-\pi \leqslant \phi \leqslant \pi$

B_w: $w_B = e^{i\phi}$, $s_w = \phi$, $\theta_w(s_w) = \phi + \frac{1}{2}\pi$.

We suppose R_z is also an interior region. The direction of B_z is counterclockwise. The direction angle is written as $\theta_z(\phi)$ and satisfies the condition $\theta_z(\pi) = \theta_z(-\pi) + 2\pi$:

$$G(w, w') = \log|1 - w/w'| + \log|1 - ww'^*| + \log|w'|$$

$$\log(dz/dw) = C - \int_{\phi'=-\pi}^{\pi} \log(1 - we^{-i\phi'})\, d\theta_z(\phi')/\pi,$$

$$-\tfrac{1}{2}\pi \leqslant \arg(1 - we^{-i\phi'}) \leqslant \tfrac{1}{2}\pi.$$

(Note that $\int \log(1 - we^{i\phi'})\, d\theta_w(\phi') = \int_{-\pi}^{\pi} \log(1 - we^{i\phi'})\, d\phi' = 0$.) For w on the boundary, $w = e^{i\phi}$, an alternative is the following: Apply Eqs. (3.2) and (3.3) with ϕ, ϕ' in place of u, u'. Then

$$\log dz/d\phi = C' - \int_{\phi'=-\pi}^{\pi} \log \sin \tfrac{1}{2}(\phi - \phi')\, d\theta_z(\phi')/\pi,$$

$$\begin{aligned} \arg(\sin \tfrac{1}{2}(\phi - \phi')) &= 0 \quad \text{if} \quad \phi > \phi', \\ &= \pi \quad \text{if} \quad \phi < \phi', \\ \operatorname{Im}(C') &= \theta_z(\pi). \end{aligned}$$

The constants are related by

$$\begin{aligned} C &= \operatorname{Re}(C') + 2\log 2 + i \int_{-\pi}^{\pi} [\theta_z(\phi') - \tfrac{1}{2}\pi]\, d\phi'/(2\pi) \\ &= \operatorname{Re}(C') + 2\log 2 + \arg dz(0)/d\phi. \end{aligned}$$

A corner of B_z at $\phi = \pm\pi$ can be handled by shifting the integration to $-\pi + \varepsilon \leqslant \phi' \leqslant \pi + \varepsilon$ and taking $\varepsilon \to 0$.

The real part of the $\log dz/d\phi$ equation was derived earlier by Noble [9] by a method which, like the present one, does not use a limiting approximation by polygons.

5. *Exterior of unit disc.*

R_w: $w = \rho e^{i\phi}$, $1 < \rho < \infty$, $-\pi \leqslant \phi \leqslant \pi$

B_w: $w_B = e^{i\phi}$, $s_w = -\phi$, $\theta_w(s_w) = \phi - \frac{1}{2}\pi$.

We suppose R_z is also an exterior region. The direction of B_z is clockwise. The direction angle is written as $\theta_z(\phi)$, and satisfies the condition $\theta_z(\pi) = \theta_z(-\pi) + 2\pi$:

$$G(w, w') = \log|1 - w'/w| + \log|1 - (ww'^*)^{-1}| - \log|w'|$$

$$\log(dz/dw) = C - \int_{\phi'=\pi}^{-\pi} \log(1 - e^{i\phi'}/w)\, d\theta_z(\phi')/\pi,$$

$$-\tfrac{1}{2}\pi \leqslant \arg(1 - e^{i\phi'}/w) \leqslant \tfrac{1}{2}\pi.$$

As an alternative for w on the boundary, $w = e^{i\phi}$:

$$\log(dz/d\phi) = C' - \int_{\phi'=\pi}^{-\pi} \log \sin \tfrac{1}{2}(\phi - \phi')\, d\theta_z(\phi')/\pi,$$

$$\arg(\sin \tfrac{1}{2}(\phi - \phi')) = 0 \quad \text{if} \quad \phi > \phi',$$
$$= -\pi \quad \text{if} \quad \phi < \phi',$$
$$\mathrm{Im}(C') = \theta_z(\pi).$$

The constants are related by

$$C = \mathrm{Re}(C') - 2\log 2 + i \int_{-\pi}^{\pi} [\theta_z(\phi') - \tfrac{1}{2}\pi]\, d\phi'.$$

6. *Periodic geometry in the upper half plane.* Let $z = z(w)$ map the upper half w-plane to the half-infinite region $\bar{R}_z$ above a boundary $\bar{B}_z$ which is periodic in x with $(-L, L)$ as the fundamental period. We can normalize as follows:

R_w: $-\pi \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant \infty$. The points $-\pi + iv$ and $\pi + iv$ are identified as the same point.

B_w: $w_B = u$, $s_w = u$, $\theta_w(s_w) = 0$

R_z, B_z: Let B_z be the image of B_w with z_w normalized so that the abscissa for B_z is on $(-L, L)$. R_z is then the image of R_w.

The map is of the form $z(w) = (L/\pi)w + iF(w)$, where $F(w)$ is periodic and analytic in the upper half plane; it has a Fourier expansion

$$F = F(\infty) + \sum_{n=1}^{\infty} a_n e^{inw}.$$

Then

$$\log(dz/dw) \sim \log(L/\pi) + O(e^{-v}) \qquad \text{for} \quad v \to \infty$$

$$G(w, w') = \log|1 - e^{i(w-w')}| + \log|1 - e^{i(w-w'^*)}| - v'$$

$$\log(dz/dw) = C - \int_{u'=-\pi}^{\pi} \log(1 - e^{i(w-u')})\, d\theta_z(u')/\pi,$$

$$-\tfrac{1}{2}\pi \leqslant \arg(1 - e^{i(w-u')}) \leqslant \tfrac{1}{2}\pi,$$

$$C = \lim_{v\to\infty} \log(dz/dw) = \log(L/\pi).$$

By convention on the corners at $u = \pm\pi$: Take the integration interval as $(-\pi + \varepsilon, \pi + \varepsilon)$ with $\varepsilon \to 0$. Then a corner is associated with a jump $\Delta\theta_z(u)$ at $u = \pi$, but not at $u = -\pi$. More explicitly, when there is a corner at $\pm\pi$ and $f(u)$ is smooth,

$$\int_{-\pi}^{\pi} f(u')\, d\theta_z(u') \qquad \text{means} \qquad \int_{-\pi+\varepsilon}^{\pi-\varepsilon} f(u')\, d\theta_z(u') + f(\pi)\, \Delta\theta_z(\pi).$$

The following sum rules apply: $\int_{-\pi}^{\pi} d\theta_z(u')=0$ and $\int_{-\pi}^{\pi} \theta_z(u')\,du'=0$. The latter follows from $\int_{B_w} S_f(w)\,dw=0$. As an alternative for the boundary, $w=u$:

$$\log(dz/du)=C-\int_{u'=-\pi}^{\pi} \log\sin\tfrac{1}{2}(u-u')\,d\theta_z(u')/\pi,$$

$$\arg\sin\tfrac{1}{2}(u-u')=0 \quad \text{if} \quad u>u',$$
$$=\pi \quad \text{if} \quad u<u',$$
$$C=\log(L/\pi)+i\theta_z(\pi+\varepsilon), \qquad \varepsilon\to 0.$$

7. *Even periodic geometry in the upper half plane.* This follows case 6 above, with $\bar{B}_z$ symmetric under $x\to -x$. Then under $u\to -u$, $z(w)-(L/\pi)w$ is even, and $\theta_z(u)$ is odd. We can set, for any $f(u)$,

$$\int_{-\pi}^{\pi} f(u')\,d\theta_z(u')=\int_0^{\pi} [f(u')+f(-u')]\,d\theta_z(u');$$

R_w: $0\leqslant u\leqslant \pi$, $0<v<\infty$,

$$\log(dz/dw)=\log(L/\pi)-\int_{u'=0}^{\pi} \log(1-2e^{iw}\cos u'+e^{2iw})\,d\theta_z(u')/\pi,$$
$$-\pi<\arg(1-2e^{iw}\cos u'+e^{2iw})<\pi.$$

For the alternative for the boundary, $w=u$:

A form which explicitly takes into account the corners at $u=0$ and $u=\pi$, if there are any, is

$$\log(dz/du)=C-\int_{\varepsilon}^{\pi-\varepsilon} \log(\cos u-\cos u')\,d\theta_z(u')/\pi,$$

$$\arg(\cos u'-\cos u)=0 \text{ or } \pi,$$
$$C=\log(2L/\pi)+i\theta_z(\pi+\varepsilon)$$
$$-\log(\sin\tfrac{1}{2}u)\,\Delta\theta_z(0)/\pi-\log(\cos\tfrac{1}{2}u)\,\Delta\theta_z(\pi)/\pi.$$

8. *Odd periodic geometry in the upper half plane.* This follows case 6 above, with $\bar{B}_z$ antisymmetric under $x\to -x$. Then under $u\to -u$, $z(w)-(L/\pi)w$ is odd and $\theta_z(u)$ is even. We can set, for any $f(u)$,

$$\int_{-\pi}^{\pi} f(u')\,d\theta_z(u')=\int_0^{\pi} [f(u')-f(-u')]\,d\theta_z(u').$$

There can be no corners at $u'=0$ or $\pm\pi$.

R_w: $0 \leqslant u \leqslant \pi$, $0 < v < \infty$,

$$\log(dz/dw) = \log(L/\pi) - \int_0^\pi \log \frac{1-e^{i(w-u')}}{1-e^{i(w+u')}}\, d\theta_z(u')/\pi,$$

$$-\pi < \arg \frac{1-e^{i(w-u')}}{1-e^{i(w+u')}} < \pi.$$

As an alternative for the boundary, $w=u$:

$$\log(dz/du) = C - \int_0^\pi \log \frac{\sin \frac{1}{2}\,(u-u')}{\sin \frac{1}{2}\,(u+u')}\, d\theta_z(u')/\pi,$$

$$\arg \frac{\sin \frac{1}{2}\,(u-u')}{\sin \frac{1}{2}\,(u+u')} = 0 \text{ or } \pi,$$

$$C = \log(L/\pi) + i\theta_z(\pi).$$

9. *Periodic channel.*

R_w: $-\pi \leqslant u \leqslant \pi$, $0 < v < h$

$B_w^{(1)}$: $w_B = u$, $s_w = u$, $\theta_w^{(1)}(s_w) = 0$

$B_w^{(2)}$: $w_B = u + ih$ with $h > 0$, $s_w = \pi - u$, $\theta_w^{(2)}(s_w) = -\pi$.

Consider a channel in the z plane bounded below by the curve $y = \hat{y}^{(1)}(x)$ and bounded above by $y = \hat{y}^{(2)}(x)$. Let both boundary curves be periodic in x with fundamental period $(L, -L)$. Let R_z be a "fundamental period" of this channel such that its lower boundary $B_z^{(1)}$ is the image, under the map $z = z(w)$, of $B_w^{(1)}$, and $B_z^{(1)}$ extends over $-L \leqslant x \leqslant L$. The upper boundary $B_z^{(2)}$, which is the image of $B_w^{(2)}$, extends over an interval of equal length, $-L + x_0 \leqslant x \leqslant L + x_0$, with x_0 to be determined concurrently with the map. The directions of $B^{(1)}$, $B^{(2)}$ are $\theta^{(1)}(u)$, $\theta^{(2)}(u)$, respectively.

Both $z(w) - (L/\pi)w$ and $\log(dz/dw)$ are analytic and periodic functions of w in the w-channel, with period 2π. If the circuit integral of $z(w) - (L/\pi)w$ around R_w is equated to zero, the imaginary part of this relation yields

$$h = \int_{-\pi}^{\pi} [\hat{y}^{(2)}(u) - \hat{y}^{(1)}(u)]\, du/(2\pi),$$

so the height of the w-channel must also be determined concurrently with the map. Alternatively, h can be fixed in advance, and the scale size L of the z-plane periodicity taken as the unknown. The same treatment of $\log(dz/dw)$ yields

$$\int_{-\pi}^{\pi} \theta_z^{(1)}(u)\, du = \int_{-\pi}^{\pi} [\theta_z^{(2)}(u) - \pi]\, du,$$

which can serve as a diagnostic check on computed values of $\theta_z^{(1)}(u)$ and $\theta_z^{(2)}(u)$.

In the formulas below, ϑ_0 and ϑ_1 are Jacobi Theta functions with parameter $\tau = ih/\pi$ and some $q = e^{i\pi\tau} = e^{-h}$. See Appendix B for a listing of the properties of these functions and verification of the formulas for the present case:

$$G(w, w') = \log \left| \vartheta_1 \left(\frac{w - w'}{2} \right) \right| + \log \left| \vartheta_1 \left(\frac{w - w'^*}{2} \right) \right|,$$

$$\log(dz/dw) = C - \int_{u'=\pi}^{\pi} \log \vartheta_1 \left(\frac{w - u'}{2} \right) d\theta_z^{(1)}(u')/\pi$$

$$- \int_{u'=\pi}^{-\pi} \log \vartheta_1 \left(\frac{w - u' - ih}{2} \right) d\theta_z^{(2)}(u')/\pi.$$

Boundary equations: For z on $B_z^{(1)}$,

$$\log dz/dw = C^{(1)} - \int_{u'=-\pi}^{\pi} \log \vartheta_1 \left(\frac{u - u'}{2} \right) d\theta_z^{(1)}(u')/\pi$$

$$- \int_{u'=\pi}^{-\pi} \log \vartheta_0 \left(\frac{u - u'}{2} \right) d\theta_z^{(2)}(u')/\pi,$$

$$\arg \vartheta_1 \left(\frac{u - u'}{2} \right) = 0 \qquad \text{if} \quad u > u',$$

$$= \pi \qquad \text{if} \quad u < u',$$

$$\arg \vartheta_0 \left(\frac{u - u'}{2} \right) = 0, \qquad \operatorname{Im} C^{(1)} = \theta_z^{(1)}(\pi).$$

For z on $B_z^{(2)}$,

$$\log dz/dw = C^{(2)} - \int_{u'=-\pi}^{\pi} \log \vartheta_0 \left(\frac{u - u'}{2} \right) d\theta_z^{(1)}(u')/\pi$$

$$- \int_{u'=\pi}^{-\pi} \log \vartheta_1 \left(\frac{u - u'}{2} \right) d\theta_z^{(2)}(u')/\pi,$$

$$\arg \vartheta_1 \left(\frac{u - u'}{2} \right) = 0 \qquad \text{if} \quad u > u',$$

$$= -\pi \qquad \text{if} \quad u < u',$$

$$\arg \vartheta_0 \left(\frac{u - u'}{2} \right) = 0, \qquad \operatorname{Im} C^{(2)} = \theta_z^{(2)}(\pi) - \pi.$$

The relations among the constant are (from Appendix B):

$$C = C^{(1)} + i \left[\theta_z^{(2)}(\pi) - \int_{-\pi}^{\pi} \theta_z^{(2)}(u)\, du/(2\pi) \right]$$

$$= C^{(2)} + i \left[\theta_z^{(1)}(\pi) - \int_{-\pi}^{\pi} \theta_z^{(1)}(u)\, du/(2\pi) \right].$$

Corners at the ends of $B_z^{(1)}$, $B_z^{(2)}$ may be handled as in case 6, above.

Floryan [6] has calculated maps of general periodic channels to a standard channel by repeating the periodic pattern n times in R_z and applying the procedure for a nonperiodic channel and then letting n increase until sufficient convergence is obtained. This may be shown to be equivalent to approximating the ϑ functions by the first n terms of their representations as infinite products. We expect to develop the analysis and calculational methods for the periodic channel using the present formulation in a subsequent study.

10. *Annular region.* The transformations $W = e^{i(\pi - w)}$, $w = i \log W + \pi$ define a conformal map between the fundamental period R_w of the periodic channel in case 9 above and the annulus $1 < |W| < b$ in a complex W plane, with $b = e^h$. (In this map, corresponding points $-\pi + iv$ and $\pi + iv$ in the w plane are identified as the same interior point of R_w; i.e., R_w is a projective rectangle.)

The conformal mapping between the W-annulus and a doubly connected region of a complex Z-plane can be addressed directly, but it seems simpler, conceptually and notationally, to set $Z = e^{i(\pi - z)}$ and treat the mapping between z and w according to case 9. The doubly periodic nature of Green's function has a clearer geometric expression and the theta functions have arguments of the form $\frac{1}{2}(w - w')$ rather than $\frac{1}{2} i \log(W/W')$.

Appendix A

We wish to verify that the integral of Eq. (3.6) of this text,

$$F = \int_{u' = -\infty}^{\infty} \log |u - u'| \, d\theta_z(u')$$

behaves, in the limits $u \to \pm\infty$, like

$$F_0 = \log |u| \int d\theta_z(u').$$

under appropriate conditions on $d\theta_z/du$ for large u. First set $F = F_0 + K$, so that

$$K = \int_{u' = -\infty}^{\infty} \log |1 - u'/u| \, d\theta_z(u').$$

We must show $K \to 0$ as $|u| \to \infty$ under the conditions (3.8) for Case 1 of Section III.D and under the conditions (3.10) for Case 2. We treat only Case 2 explicitly here. The convergence of K to zero is stronger for Case 1 and can be handled along the same lines.

To begin, divide the integration interval into 3 parts

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{-u_0} + \int_{-u_0}^{u_0} + \int_{u_0}^{\infty},$$

where $|u_0|$ is so large that the asymptotic expressions in (3.10) for $d\theta_z/du$ can be used in the first and third parts. We look in detail only at the third part; the argument for the first part will be essentially the same, and the contribution to K from the interval $(-u_0, u_0)$ clearly goes to zero with $|u| \to \infty$, as u_0 is fixed. The third part of the K integral, for large enough u_0 and the conditions of Case 2, can be written, apart from a constant factor which we ignore,

$$\int_{u_0}^{\infty} \log|1-u'/u| \frac{du'}{u'(\log u')^{1+a}}.$$

Divide the integration interval (u_0, ∞) into four parts

$$\int_{u_0}^{\infty} = \int_{u_0}^{\sqrt{u}} + \int_{\sqrt{u}}^{1/2u} + \int_{1/2u}^{2u} + \int_{2u}^{\infty}$$

and let their respective contributions to K be called K_1, K_2, K_3, K_4.

First,

$$\begin{aligned} |K_1| &\leqslant \int_{u_0}^{\sqrt{u}} |\log(1-1/\sqrt{u})| \frac{du'}{u'(\log u')^{1+a}} \\ &\leqslant |\log(1-1/\sqrt{u})| \, [(\log u_0)^{-a} - (\log \sqrt{u})^{-a}]/a, \end{aligned}$$

so $|K_1|$ behaves like $u^{-1/2}$ as $u \to \infty$.

Second, in the K_2 integral, $|\log(1-u'/u)| \leqslant |\log(\frac{1}{2})|$, so $|K_2| \sim (\log \sqrt{u})^{-a} - (\log \frac{1}{2} u)^{-a}$ as $u \to \infty$.

Third,

$$\begin{aligned} |K_3| &\leqslant |\log \tfrac{1}{2} u|^{-a} \int_{1/2u}^{2u} \log|1-u'/u| \, du'/u' \\ &\leqslant |\log \tfrac{1}{2} u|^{-a} \int_{1/2}^{2} \log|1-x| \, dx/x, \end{aligned}$$

and the x-integral is finite.

Fourth, starting from $2u < u'$ and $1 < u$, we have $u < u' - u < u' < u'u$, so in the K_4 integral,

$$0 < \log(u'-u) - \log u < \log(u'u) - \log u = \log u'.$$

Hence,

$$|K_4| < \int_{2u}^{\infty} \frac{du'}{u'(\log u')^{a}} = (\log 2u)^{1-a}/(a-1).$$

Therefore $K = K_1 + K_2 + K_3 + K_4 \to 0$ as $u \to \infty$ because $a > 1$ for Case 2.

APPENDIX B

This appendix provides technical details that aid in verification and utilization of the equations for the periodic channel, Case 9 of Section IV. Variables $w = u + iv$ and $w' = u' + iv'$ are assumed to be in R_w for that case, or on its boundary, i.e., $-\pi \leqslant u,\ u' \leqslant \pi,\ 0 \leqslant v,\ v' \leqslant h$. The nome q and second period $\pi\tau$ of the relevant Jacobi Theta functions, primarily $\vartheta_0(z)$ and $\vartheta_1(z)$, are related to the height of R_w by

$$q = e^{i\pi\tau} = e^{-h}, \qquad \tau = ih/\pi. \tag{B.1}$$

1. *Useful Properties of Theta Functions*

Series and products.

$$\vartheta_0(z) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2nz) \tag{B.2}$$

$$\vartheta_1(z) = 2\sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)z \tag{B.3}$$

$$\vartheta_0(z) = P(q) \prod_{n=1}^{\infty} (1 - 2q^{2n+1}\cos 2z + q^{4n-2}) \tag{B.4}$$

$$\vartheta_1(z) = 2P(q)\, q^{1/4} \sin z \prod_{n=1}^{\infty} (1 - 2q^{2n}\cos 2z + q^{4n}) \tag{B.5}$$

$$\log \vartheta_0(z) = \log P(q) - 2\sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} \frac{\cos(2nz)}{n} \tag{B.6}$$

$$\log \vartheta_1(z) = \log \sin z + \log(2P(q)q^{1/4}) - 2\sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \frac{\cos(2nz)}{n}, \tag{B.7}$$

where

$$P(q) = \prod_{n=1}^{\infty} (1 - q^{2n}), \qquad P^3(q) = \sum_{n=0}^{\infty} (-1)^n (2n+1)\, q^{n^2+n}. \tag{B.8}$$

Zeros (n, m are integers).

$$\vartheta_0(z) = 0 \quad \text{at} \quad z = \tfrac{1}{2}\pi\tau + n\pi + m\pi\tau \tag{B.9}$$

$$\vartheta_1(z) = 0 \quad \text{at} \quad z = n\pi + m\pi\tau. \tag{B.10}$$

Shifts.

$$\vartheta_1(z \pm \tfrac{1}{2}\pi\tau) = \pm i q^{-1/4} e^{\mp iz}\, \vartheta_0(z) \tag{B.11}$$

$$\vartheta_0(z+\pi) = \vartheta_0(z), \qquad \vartheta_1(z+\pi) = -\vartheta_1(z) \tag{B.12}$$

$$\vartheta_i(z) = -q e^{2iz}\, \vartheta_i(z + \pi\tau), \qquad i = 0, 1. \tag{B.13}$$

Reciprocity (set $\tau' = -1/\tau = i\pi/h$, $\log q' = \pi^2/\log q$).

$$\vartheta_0(z, \tau) = (-i\tau')^{1/2}(q')^{z^2/\pi^2}\, \vartheta_2(z\tau', \tau') \tag{B.14}$$

$$\vartheta_1(z, \tau) = -i(-i\tau')^{1/2}(q')^{z^2/\pi^2}\, \vartheta_1(z\tau', \tau') \tag{B.15}$$

$$P^3(q) = (q'/q)^{1/4}(-i\tau')^{3/2}\, P^3(q') \tag{B.16}$$

with

$$\vartheta_2(z, \tau) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2}\cos(2n+1)z. \tag{B.17}$$

The above equations are all (or nearly all) that one needs to know about theta functions for theory and computation of SCEs for periodic channels. For $h \geqslant \pi$ and hence $q \leqslant e^{-\pi} < 0.0433$, and for $z = \frac{1}{2}(w - w')$, the first four terms of the sums in (B.2) and (B.3) provide machine accuracy (14 figures), or better, for needed values of the theta functions. For $h \leqslant \pi$, one applies (B.14), (B.15) and four terms of the series for $\vartheta_2(z\tau', \tau')$ and $\vartheta_1(z\tau', \tau')$ are sufficient for 14 figures.

2. *Green's Function*

We now verify that

$$G(w, w') = \log\left|\vartheta_1\left(\frac{w-w'}{2}\right)\right| + \log\left|\vartheta_1\left(\frac{w-w'^*}{2}\right)\right| \tag{B.18}$$

may be taken as Green's function for R_w, with constant normal-derivative boundary conditions.

A heuristic derivation, not offered here, might proceed by adding a series of harmonics periodic in w to $\log|\sin\frac{1}{2}(w-w')|$ to meet boundary requirements; one would arrive at (B.18) via formula (B.7). To confirm (B.18) directly, we observe:

(a) $G(w, w')$, as given by (B.18) is periodic in w with period 2π, symmetric in w and w', and analytic in R_w except where the theta functions vanish.

(b) For w, w' in R_w, (B.10) implies that $\vartheta_1((w-w')/2)$ vanishes only at $w = w'$; $\vartheta_1((w-w'^*)/2)$ does not vanish. By (B.7), the singularity of $G(w, w')$ at $w = w'$ is like $\log|\sin\frac{1}{2}(w-w')|$ and hence like $\log|w-w'|$; so G satisfies

$$\nabla_w^2\, G(w, w') = 2\pi\, \delta(u-u')\, \delta(v-v').$$

(c) Set $\log|\vartheta_1((w-w')/2)| = f(v)$, for short. Then

$$\log\left|\vartheta_1\left(\frac{w-w'^*}{2}\right)\right| = \log\left|\vartheta_1\left(\frac{w^*-w'}{2}\right)\right| = f(-v).$$

Also, by (B.13), with $z = \frac{1}{2}(w^* - w')$ and $\pi\tau = ih$,

$$\log\left|\vartheta_1\left(\frac{w^*-w'}{2}\right)\right| = f(2h-v) + v - v' - h.$$

Then, on boundary $B_w^{(1)}$ of R_w,

$$\frac{\partial G}{\partial n} = \left\{ -\frac{\partial}{\partial v} [f(v) + f(-v)] \right\}_{v=0} = 0,$$

and on boundary $B_w^{(2)}$ of R_w,

$$\frac{\partial G}{\partial n} = \left\{ \frac{\partial}{\partial v} [f(v) + f(2h - v) + v - v' - h] \right\}_{v=h} = 1.$$

This completes the verification of (B.18).

3. *Boundary Relations*

Applying Green's theorem as described in Section II, one obtains the SCE for the map $z = z(w)$ without subtraction in the form

$$\log(dz/dw) = C + Q_1(w) + Q_2(w),$$

$$Q_1(w) = -\int_{u'=-\pi}^{\pi} \log \vartheta_1\left(\frac{w - u'}{2}\right) d\theta_z^{(1)}(u')/\pi,$$

$$Q_2(w) = -\int_{u'=\pi}^{-\pi} \log \vartheta_1\left(\frac{w - u' - ih}{2}\right) d\theta_z^{(2)}(u')/\pi.$$

Let w approach boundary $B_w^{(1)}$, i.e., $w = u + i\varepsilon$, $\varepsilon \to 0$. Then

$$Q_1(w) \to Q_1(u) = -\int_{u'=-\pi}^{\pi} \log \vartheta_1\left(\frac{u - u'}{2}\right) d\theta_z^{(1)}(u')/\pi,$$

where, noting (B.5) or (B.7) above,

$$\begin{aligned}
\operatorname{Im} \log \vartheta_1\left(\frac{u - u'}{2}\right) &= \operatorname{Im} \lim_{\varepsilon \to 0} \log \sin \tfrac{1}{2}\,(u + i\varepsilon - u') \\
&= \operatorname{Im} \lim_{\varepsilon \to 0} \log(u - u' + i\varepsilon) \\
&= 0 \qquad \text{if} \quad u > u', \\
&= \pi \qquad \text{if} \quad u < u'.
\end{aligned}$$

To separate the real and imaginary parts of $Q_2(w) = Q_2(u)$ for w on $B_w^{(1)}$, apply (B.11) in the form

$$\vartheta_1\left(\frac{u - u' - ih}{2}\right) = e^{(1/2)i(u - u' - \pi)}\, e^{(1/4)h}\, \vartheta_0\left(\frac{u - u'}{2}\right)$$

to get

$$Q_2(w) \to Q_2(u) = -\int_{u'=\pi}^{-\pi} \log \vartheta_0\left(\frac{u-u'}{2}\right) d\theta_z^{(2)}(u')/\pi$$

$$-\int_{u'=\pi}^{-\pi} \left[\tfrac{1}{2} i(u-u'-\pi) + \tfrac{1}{4} h\right] d\theta_z^{(2)}(u')/\pi$$

$$= -\int_{u'=\pi}^{-\pi} \log \vartheta_0\left(\frac{u-u'}{2}\right) d\theta_z^{(2)}(u')/\pi$$

$$-i\theta_z^{(2)}(\pi) + i\int_{-\pi}^{\pi} \theta_z^{(2)}(u')\, du'/(2\pi).$$

For the last equation, the periodicity of $\theta_z^{(2)}$ and an integration by parts were utilized. The $\log \vartheta_0$ term is real for $-\pi \leqslant u, u' \leqslant \pi$ because ϑ_0 is positive, as can be inferred from (B.4).

The limiting forms of $Q_1(w)$ and $Q_2(w)$ as w goes to $B_w^{(2)}$, i.e., $w = u + ih - i\varepsilon$, $\varepsilon \to 0$, are processed in the same manner. We find

$$Q_2(w) \to Q_2(u+ih) = -\int_{u'=\pi}^{-\pi} \log \vartheta_1\left(\frac{u-u'}{2}\right) d\theta_z^{(2)}(u')/\pi,$$

where

$$\operatorname{Im} \log \vartheta_1\left(\frac{u-u'}{2}\right) = \operatorname{Im} \lim_{\varepsilon \to 0} (u-u'-i\varepsilon)$$

$$= 0 \qquad \text{if} \quad u > u',$$

$$= -\pi \qquad \text{if} \quad u < u'.$$

And, applying (B.11) in the form

$$\vartheta_1\left(\frac{u+ih-u'}{2}\right) = e^{(1/2)i(-u+u'+\pi)}\, e^{(1/4)h}\, \vartheta_0\left(\frac{u-u'}{2}\right),$$

we get

$$Q_1(w) \to Q_1(u+ih) = -\int_{u'=-\pi}^{\pi} \log \vartheta_0\left(\frac{u-u'}{2}\right) d\theta_z^{(1)}(u')/\pi$$

$$-i\theta_z^{(1)}(\pi) + i\int_{-\pi}^{\pi} \theta_z^{(1)}(u')\, du'/(2\pi).$$

These all relations are summarized in Section IV.9.

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